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Subduction coefficients of Brauer algebras and Racah coefficients of O(n) and $Sp(2m)$: II. Racah coefficients

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Abstract. Racah coefficients of O(n) and $Sp(2m)$ are derived from subduction coefficients of Brauer algebras $D_f(n)$ by using the Schur–Weyl duality relation between $D_f(n)$ and O(n) or $Sp(2m)$. It is found that there are two types of Racah coefficients according to irreps of O(n) or $Sp(2m)$ with or without trace contraction. It is proved that Racah coefficients with no trace contraction in the irreps are trivial and the same as those of unitary groups $U(n)$, which are rank n-independent, and those with trace contraction usually are n-dependent. Racah coefficients with trace contraction for the resulting irreps $[n_1, n_2, n_3, \dot{0}]$ with $\sum_{i=1}^{3} n_i \leqslant 3$ are tabulated.

1. Introduction

Racah coefficients of classical Lie groups are useful in many branches of physics, especially in atomic spectroscopy, nuclear structure and particle physics. There has been a great deal of work on this topic since the pioneering work of Wigner [1] and Racah [2]. These coefficients or so-called $6j$ symbols by a different definition were first initiated from recoupling problems in angular momentum theory, for which the underlying group is $SO(3)$. Then, the same problem was also found to be of importance in disparate fields of physics. There are many approaches to the Racah coefficients; the literature is now awash with different expressions for certain kinds of $3nj$ symbols. However, analytical expressions are difficult to come by for the general Lie group, mainly because there is a multiplicity problem in the reduction of Kronecker products of pairs of irreps. Some missing labels need to be added in, for which a procedure is often difficult to do systematically. Usually Racah coefficients can be obtained by using a knowledge of a few simple case to get others through the extension of the Biedenharn–Elliot sum rule. This bootstrap method was developed by Bickerstaff and Wybourne [3], Searle and Butler [4]. There are also many other methods. For example, generating functions can be used in some special cases [5], isoscalar factors can be constructively used in some cases [6], and we sometimes use the mathematical structure inherent in a particular physical problem [7–10]. We should emphasize the works of Kramer [11], and Chen *et al* [12]. They used the Schur–Weyl duality relation between the symmetric group S_f and the unitary group $U(n)$, which enabled them to derive $U(n)$ Racah coefficients from subduction coefficients of S_f. One can calculate these coefficients once and for all because, in this case, the Racah coefficients of $U(n)$ only depend on irreps, not on rank n. Extensive tables of $U(n)$ Racah coefficients were thus made by using this method [12].

Algebraic expressions of $SO(n)$ Racah coefficients were discussed for some special cases by Judd *et al* [9, 10], and Alisăuskas [6]. In [10], Judd *et al* also claimed their $SO(n)$ results with conjugation of irreps apply to $Sp(-n)$ as well. The negative-dimensional groups were first discussed in the physics literature by Čvitanović and Kennedy [13], and further by Dunne [14]. The isomorphism between O(n) and $Sp(-n)$ with conjugation of the same irrep were also proved in mathematics with the help of Brauer's centralizer algebras [15–17].

In our previous paper, we have outlined a method for evaluating subduction coefficients of Brauer algebras $D_f(n)$. In the following, we will use the results to derive Racah coefficients of O(n) and $Sp(2m)$ with the help of the Schur–Weyl duality relation between $D_f(n)$ and O(n) or $Sp(2m)$. In section 2, we will discuss the Schur–Weyl duality relation and present a general formula for evaluating Racah coefficients. In section 3, we will discuss some general properties of these coefficients. In section 4, we will list some rank-dependent Racah coefficients for the resulting irreps $[n_1, n_2, n_3, \dot{0}]$ with $\sum_{i=1}^{3} n_i \leqslant 3$.

2. Evaluation of $SO(n)$ and $Sp(2m)$ Racah coefficients

O(n) and $Sp(2m)$ Racah coefficients, the so-called U coefficients are simply a generalization of the $SO(3)$ Racah coefficients, which are the elements of a unitary matrix between bases with two different coupling orders of three irreps v_1, v_2, and v_3:

$$|(v_1v_2)v_{12}, v_3; vw\rangle^{t_{12}t} = \sum_{v_{23}t_{23}t'} U(v_1v_2vv_3; v_{12}v_{23})^{t_{12}t}_{t_{23}t'} |v_1(v_2v_3)v_{23}; vw\rangle^{t_{23}t'} \tag{1}$$

where four multiplicity labels appeared, $t_{12} = 1, 2, \ldots, \{v_1v_2v_{12}\}$, $t = 1, 2, \ldots, \{v_{12}v_3v\}$, $t' = 1, 2, \ldots, \{v_1v_{23}v\}$. Sometimes one needs to use $6j$ symbols, which, in analogy to that of Jahn [18] for $SO(3)$, is defined in terms of U coefficients by

$$\begin{Bmatrix} v_1 & v_2 & v_{12} \\ v_3 & v & v_{23} \end{Bmatrix}^{t_{12}t}_{t_{23}t'} = [P_{v_{12}}(n)P_{v_{23}}(n)]^{-1/2} U(v_1v_2vv_3; v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{2}$$

where $P_\mu(n)$ is the dimension of O(n) or $Sp(n)$ for the irrep μ. The U coefficients satisfy the following unitarity conditions:

$$\begin{aligned} &\sum_{v_{23}t_{23}t'} U(v_1v_2vv_3; v_{12}v_{23})^{t_{12}t}_{t_{23}t'} U(v_1v_2vv_3; \bar{v}_{12}v_{23})^{\rho_{12}\rho}_{t_{23}t'} = \delta_{t_{12}\rho_{12}}\delta_{t\rho}\delta_{v_{12}\bar{v}_{12}} \\ &\sum_{t_{12}v_{12}t} U(v_1v_2vv_3; v_{12}v_{23})^{t_{12}t}_{t_{23}t'} U(v_1v_2vv_3; v_{12}\bar{v}_{23})^{t_{12}t}_{\rho_{23}\rho'} = \delta_{t_{23}\rho_{23}}\delta_{t'\rho'}\delta_{v_{23}\bar{v}_{23}}. \end{aligned} \tag{3}$$

From the early work of Brauer [15] and recent study [16, 17] one knows that there is an important relation, the so-called Schur–Weyl duality relation between the Brauer algebra $D_f(n)$ and O(n) or $Sp(2m)$. If G is the orthogonal group O(n) or the sympletic group $Sp(2m)$, the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer's $D_f(n)$ or $D_f(-2m)$, respectively. We also need a special class of Young diagrams, the so-called n-permissible Young diagrams defined in [17]. A Young diagram $[\lambda]$ is said to be n-permissible if $P_\mu(n) \neq 0$ for all subdiagrams $[\mu] \leqslant [\lambda]$, where the subdiagrams $[\mu]$ can be obtained from $[\lambda]$ by taking away appropriate boxes. In this paper, we only need to discuss the integer n case. A Young diagram $[\lambda]$ is n-permissible if and only if:

(i) Its first two columns contain at most n boxes for n positive.

(ii) It contains at most m columns for $n = -2m$ a negative even integer.

(iii) Its first two rows contain at most $2 - n$ boxes for n odd and negative.

If these conditions are satisfied, $B_f(n)$ is isomorphic to $B_f(\mathrm{O}(n))$ for n positive, to $B_f(\mathrm{O}(2-n))$ for n negative and odd, and to $B(Sp(2m))$ for $n = -2m < 0$. In the following, we always assume that all irreps to be discussed are n-permissible with $n \leqslant f-1$ for $n > 0$ or $-n \leqslant f-1$ for negative n. The latter condition implies that $D_f(n)$ is to be considered semisimple.

Hence, an irrep of $B_f(\mathrm{O}(n))$ or $B_f(Sp(2m))$ is the same irrep of $\mathrm{O}(n)$ or $Sp(2m)$. But the labelling schemes of $B(G)$ and G are different. The former is labelled by its Brauer algebra indices, while the latter is labelled by its tensor components. This is the so-called Schur–Weyl duality relation between $B_f(G)$ and G, where $G = \mathrm{O}(n)$ or $Sp(2m)$.

$B_f(G)$ invariants defined by

$$U(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} = \sum_{\rho_{12}\rho_{23}\rho} \left\langle \begin{matrix} v \\ \rho \end{matrix} \middle| v,\, \begin{matrix} tv_{12} & v_3 \\ \rho_{12} & \rho_3 \end{matrix} \right\rangle \left\langle \begin{matrix} v_{12} \\ \rho_{12} \end{matrix} \middle| v_{12},\, \begin{matrix} t_{12}v_1 & v_2 \\ \rho_1 & \rho_2 \end{matrix} \right\rangle \times \left\langle \begin{matrix} v \\ \rho \end{matrix} \middle| v,\, \begin{matrix} t'v_1 & v_{23} \\ \rho_1 & \rho_{23} \end{matrix} \right\rangle \left\langle \begin{matrix} v_{23} \\ \rho_{23} \end{matrix} \middle| v_{23},\, \begin{matrix} t_{23}v_2 & v_3 \\ \rho_2 & \rho_3 \end{matrix} \right\rangle \tag{4}$$

where

$$\left\langle \begin{matrix} v \\ \rho \end{matrix} \middle| v,\, \begin{matrix} tv_1 & v_2 \\ \rho_1 & \rho_2 \end{matrix} \right\rangle$$

is the subduction coefficient of $B_f(G)$, and the summation in (4) carried out under fixed ρ_1, ρ_2, and ρ_3, only depends on irreps v_1, v_2, v_3, v, v_{12}, v_{23}, and does not depend on sub-indices. According to the Schur–Weyl duality relation, (4) is also the U coefficients of G satisfying the unitarity condition given in (3). One can thus use (4) to calculate Racah U coefficients of $\mathrm{O}(n)$ and $Sp(2m)$ from subduction coefficients of Brauer algebras $D_f(n)$. The SDCs of $D_f(n)$ are the same as those of S_f when there is no trace contraction in the irreps. These SDCs have already been listed in [12]. While SDCs with trace contractions in the irreps considered for the reductions $D_f(n) \downarrow D_{f_1}(n) \times D_{f_2}(n)$ for $f = f_1 + f_2 \leqslant 5$ have already been given in our previous paper I. We can use these results with (4) to calculate Racah coefficients of $\mathrm{O}(n)$ or $Sp(2m)$.

3. Some properties of $\mathrm{O}(n)$ and $Sp(2m)$ Racah coefficients

It is proved in [17] that generators $\{\tilde{g}_i, \tilde{e}_i\}$ of $B_f(Sp(2m))$ are compatible with the relation for $\{-g_i, -e_i\}$ of $D_f(x)$ with $x = -2m$. Thus, there exists an isomorphism between $B_f(\mathrm{O}(n))$ and $B_f(Sp(2m))$ by making the maps from $B_f(\mathrm{O}(n))$ to $B_f(Sp(2m))$ with $g_i \to -\tilde{g}_i$, $e_i \to -\tilde{e}_i$, and $n \to -2m$. In this case an irrep $[\lambda]$ of $B_f(\mathrm{O}(-n))$ is the irrep $[\tilde{\lambda}]$ of $B_f(Sp(2m))$, where $[\tilde{\lambda}]$ is the Young diagram conjugate to $[\lambda]$. Hence, U coefficients derived from (4) for $\mathrm{O}(n)$ are also those of $Sp(-2m)$ up to a phase with the replacements: $n \to -2m$ and all irreps $v_i \to \tilde{v}_i$. That is, we only need to derive $\mathrm{O}(n)$ Racah coefficients because these coefficients are also those of $Sp(2m)$ up to a phase factor with the above mentioned replacements. This property of U coefficients of $\mathrm{O}(n)$ and $Sp(2m)$ asserts the negative dimensionality discussed in [13, 14]. Secondly, irreps of $D_f(n)$ without trace contraction are the same as those of the symmetric groups S_f. Using this fact and expressions of U coefficient given in (4), one can deduce that Racah coefficients of $\mathrm{O}(n)$ with no trace contraction for the irreps in the coupling are the same as Racah U coefficients of $U(n)$ because Racah coefficients of $U(n)$ are also U coefficients of symmetric groups with the same expression given in (4), which was first discussed by Kramer [11] and then used to derive $U(n)$ Racah coefficients extensively by Chen *et al* [12]. Hence, there are two types of $\mathrm{O}(n)$ or $Sp(2m)$ Racah U coefficients. The first type of U coefficients of $\mathrm{O}(n)$ are no trace contractions for the irreps in the coupling and the same as those of $U(n)$. This type of U coefficient is rank n-independent. The numerous tables of $U(n)$ Racah coefficients given in [12] are also type one Racah coefficients of O(n). Using the isomorphism between $B_f(O(n))$ and $B_f(Sp(2m))$, we deduce that type one Racah coefficients of O(n) and that of $Sp(2m)$ satisfy

$$\begin{aligned} U_{U(n)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} &= U_{\mathrm{O}(n)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \\ U_{U(n)}(\tilde{v}_1\tilde{v}_2\tilde{v}\tilde{v}_3;\, \tilde{v}_{12}\tilde{v}_{23})^{t_{12}t}_{t_{23}t'} &= \eta_1 U_{Sp(n\to -2m)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \end{aligned} \tag{5}$$

where the phase is chosen in the same way as that of $U(n)$. Furthermore, using symmetry property of $U(n)$ Racah coefficients

$$U_{U(n)}(\tilde{v}_1\tilde{v}_2\tilde{v}\tilde{v}_3,\, \tilde{v}_{12}\tilde{v}_{23})^{t_{12}t}_{t_{23}t'} = \eta_1 U_{U(n)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{6}$$

where η_1 is phase factor which has been determined in [12], we get the following equality for the type one Racah coefficients:

$$U_{\mathrm{O}(n)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} = U_{Sp(2m)}(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{7}$$

because Racah coefficients of $U(n)$ are rank n-independent.

The type two U coefficients of O(n) or $Sp(2m)$ with trace contractions in the irreps, however, are usually rank n-dependent. Symmetry property given by (6) is not valid for type two Racah coefficients of O(n) and $Sp(2m)$ in general. We shall list some of the Racah coefficients of this type in the next section.

Both type-one and type-two Racah coefficients have the symmetry

$$U_G(v_3v_2vv_1;\, v_{23}v_{12})^{t_{23}t'}_{t_{12}t} = \eta_2 U_G(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{8}$$

where $G = O(n)$ or $Sp(2m)$, and η_2 is equal to

$$\eta_2 = \epsilon_2(v_1v_2v_{12}t_{12})\epsilon_2(v_{12}v_3vt)\epsilon_2(v_2v_3v_{23}t_{23})\epsilon_2(v_1v_{23}vt') \tag{9}$$

which has been listed in [12] for type-one Racah coefficients and ϵ_2 is given in table 1–4 for type-two cases.

Table 1. The Phase factors $\epsilon_2(v_1v_2v)$.

$D_2(n)\times D_1(n)\downarrow D_3$				$D_3(n)\times D_1(n)\downarrow D_4$							
$[v_1]$	$[v_2]$	$[v]$	ϵ_2	$[v_1]$	$[v_2]$	$[v]$	ϵ_2	$[v_1]$	$[v_2]$	$[v]$	ϵ_2
[0]	[1]	[1	-1	[1]	[3]	[2]	$+1$	[1]	[1]	[2]	$+1$
[2]	[1]	[1]	$+1$	[1]	[21]	[2]	-1	[1]	[1]	$[1^2]$	-1
$[1^2]$	[1]	[1]	$+1$	[1]	$[1^3]$	$[1^2]$	$+1$	[1]	[21]	$[1^2]$	$+1$

Table 2. The Phase factors $\epsilon_2(v_1v_2v)$.

$D_2(n)\times D_2(n)\downarrow D_4$							
$[v_1]$	$[v_2]$	$[v]$	ϵ_2	$[v_1]$	$[v_2]$	$[v]$	ϵ_2
[2]	$[1^2]$	$[1^2]$	$+1$	[0]	$[1^2]$	$[1^2]$	-1
[2]	$[1^2]$	[2]	-1	[0]	[2]	[2]	$+1$
[2]	[2]	[2]	$+1$	$[1^2]$	$[1^2]$	$[1^2]$	-1
$[1^2]$	$[1^2]$	[2]	$+1$	[2]	[2]	$[1^2]$	$+1$

Table 3. The Phase factors $\epsilon_2(v_1 v_2 v)$.

$D_2(n) \times D_3(n) \downarrow D_5$							
$[v_1]$	$[v_2]$	$[v]$	ϵ_2	$[v_1]$	$[v_2]$	$[v]$	ϵ_2
$[1^2]$	[21]	$[1^3]$	-1	[2]	$[1^3]$	$[1^3]$	-1
$[1^2]$	$[1^3]$	$[1^3]$	$+1$	[2]	[21]	$[1^3]$	$+1$
$[1^2]$	[1]	[13]	$+1$	[2]	[21]	[3]	-1
[2]	[3]	[3]	$+1$	[3]	[0]	[3]	$+1$
$[1^2]$	[21]	[3]	$+1$	[2]	[1]	[3]	$+1$

Table 4. The Phase factors $\epsilon_2(v_1 v_2 v)$.

$D_1(n) \times D_4(n) \downarrow D_5$							
$[v_1]$	$[v_2]$	$[v]$	ϵ_2	$[v_1]$	$[v_2]$	$[v]$	ϵ_2
[1]	$[1^2]$	$[1^3]$	-1	[1]	$[1^4]$	$[1^3]$	-1
[1]	[211]	$[1^3]$	-1	[1]	[2]	[3]	$+1$
[1]	[31]	[3]	-1	[1]	[4]	[3]	$+1$

4. Tables of O(n) and $Sp(2m)$ Racah coefficients

In this section, we will list the type two Racah coefficients of O(n) for the resulting irreps $[n_1, n_2, n_3, \dot{0}]$ with $\sum_{i=1}^{3} n_i \leqslant 3$, which are derived by using (4) and SDCs of Brauer algebras given in I. From the results other Racah coefficients of O(n) can be obtained by the symmetry property given in (8), which have not been tabulated here. On the other hand,

Table 5. Racah coefficients $U_{Sp(2m)}(1\ 1^2\ 1^2\ 1;\, v_{12} v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	$[1^3]$
[0]	$-\frac{m+1}{(m-1)(2m+1)}$	$-\frac{2m}{2m+1}\sqrt{\frac{m+1}{3(m-1)}}$	$\sqrt{\frac{4(m-2)m^2}{3(2m+1)(m-1)^2}}$
[21]	$-\frac{2m}{2m+1}\sqrt{\frac{m+1}{3(m-1)}}$	$\frac{4m+1}{3(2m+1)}$	$\sqrt{\frac{4(m-2)(m+1)}{9(2m+1)(m-1)}}$
$[1^3]$	$\sqrt{\frac{4(m-2)m^2}{3(2m+1)(m-1)^2}}$	$\sqrt{\frac{4(m-2)(m+1)}{9(2m+1)(m-1)}}$	$\frac{m+1}{3(m-1)}$

Table 6. Racah coefficients $U_{\mathrm{O}(n)}(1\ 1\ 1\ 1;\, v_{12} v_{23})$.

	$v_{23}\backslash v_{12}$	[0]	[2]	$[1^2]$
[0]	$-\frac{1}{n}$	$\sqrt{\frac{(n+2)(n-1)}{2n^2}}$	$\sqrt{\frac{n-1}{2n}}$	
[2]	$-\sqrt{\frac{(n+2)(n-1)}{2n^2}}$	$\frac{n-2}{2n}$	$-\sqrt{\frac{n+2}{4n}}$	
$[1^2]$	$\sqrt{\frac{n-1}{2n}}$	$\sqrt{\frac{n+2}{4n}}$	$-\frac{1}{2}$	

Table 7. $U_{\mathrm{O}(n)}(2121;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	[30]
[0]	$\sqrt{\frac{2}{(n+2)(n-1)}}$	$-\sqrt{\frac{2(n-2)}{3(n-1)}}$	$\sqrt{\frac{n+4}{3(n+2)}}$
[2]	$\sqrt{\frac{(n+4)(n-2)}{2(n+2)(n-1)}}$	$\sqrt{\frac{n+4}{6(n-1)}}$	$\sqrt{\frac{n-2}{3(n+2)}}$
$[1^2]$	$-\sqrt{\frac{n^2}{2(n+2)(n-1)}}$	$\sqrt{\frac{n-2}{6(n-1)}}$	$\sqrt{\frac{n+4}{3(n+2)}}$

Table 8. $U_{\mathrm{O}(n)}(1221;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	[30]
[0]	$\frac{n-2}{(n+2)(n-1)}$	$-\frac{n}{n-1}\sqrt{\frac{n-2}{3(n+2)}}$	$\sqrt{\frac{2(n+4)n^2}{3(n-1)(n+2)^2}}$
[21]	$-\frac{n}{n-1}\sqrt{\frac{n-2}{3(n+2)}}$	$\frac{2n-1}{3(n-1)}$	$\sqrt{\frac{2(n+4)(n-2)}{9(n-1)(n+2)}}$
[3]	$\sqrt{\frac{2(n+4)n^2}{3(n-1)(n+2)^2}}$	$\sqrt{\frac{2(n+4)(n-2)}{9(n-1)(n+2)}}$	$\frac{n-2}{3(n+2)}$

Table 9. $U_{\mathrm{O}(n)}(1^2 1^2 1^3 1;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	$[1^2]$	[211]	$[1^4]$
[21]	$-\sqrt{\frac{(n-3)(n+2)}{3(n-2)(n-1)}}$	$\sqrt{\frac{(n-4)^2}{3(n-1)(n-2)}}$	$\sqrt{\frac{n+2}{3(n-1)}}$
[1]	$\sqrt{\frac{2}{(n-2)(n-1)}}$	$-\sqrt{\frac{(n+2)(n-3)}{2(n-1)(n-2)}}$	$\sqrt{\frac{n-3}{2(n-1)}}$
$[1^3]$	$\sqrt{\frac{2(n-3)}{3(n-2)}}$	$\sqrt{\frac{n+2}{6(n-2)}}$	$\sqrt{\frac{1}{6}}$

Table 10. $U_{\mathrm{O}(n)}(1^2 1 1^3 1^2;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[21]	$[1^3]$	[1]
[21]	$-\frac{n-7}{3(n-1)}$	$-\sqrt{\frac{2(n+2)}{9(n-1)}}$	$\sqrt{\frac{2(n+2)(n-3)}{3(n-1)^2}}$
$[1^3]$	$\sqrt{\frac{2(n+2)}{9(n-1)}}$	$\frac{2}{3}$	$\sqrt{\frac{n-3}{3(n-1)}}$
[1]	$\sqrt{\frac{2(n+2)(n-3)}{3(n-1)^2}}$	$-\sqrt{\frac{n-3}{3(n-1)}}$	$-\frac{2}{n-1}$

Racah coefficients of $Sp(2m)$ can also be obtained by using the following relation:

$$U_{Sp(2m)}(v_1 v_2 v v_3;\, v_{12}v_{23}) = \eta U_{\mathrm{O}(n\to -2m)}(\tilde{v}_1\tilde{v}_2\tilde{v}\tilde{v}_3;\, \tilde{v}_{12}\tilde{v}_{23}) \tag{10}$$

where η is an appropriate phase factor, which can be chosen as $+1$ because the unitarity conditions for $\mathrm{O}(n)$ Racah coefficients are satisfied for all n including the negative n case. One can check that n appears under the square root in the expressions of the Racah

Table 11. $U_{O(n)}(2132; v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	[3]
[1]	$\frac{2n}{(n-1)(n+2)}$	$-\sqrt{\frac{2n(n+1)(n-2)}{3(n+2)(n-1)^2}}$	$\sqrt{\frac{(n-2)(n+1)(n+6)}{3(n-1)(n+2)^2}}$
[21]	$-\sqrt{\frac{2n(n-2)(n+1)}{3(n+2)(n-1)^2}}$	$\frac{n-3}{3(n-1)}$	$\sqrt{\frac{2n(n+6)}{9(n+2)(n-1)}}$
[3]	$\sqrt{\frac{(n-2)(n+1)(n+6)}{3(n-1)(n+2)^2}}$	$\sqrt{\frac{2n(n+6)}{9(n+2)(n-1)}}$	$\frac{2n}{3(n+2)}$

Table 12. $U_{O(n)}(2231; v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[4]	[31]	[20]
[3]	$\sqrt{\frac{n(n-2)}{6(n+2)(n+4)}}$	$\sqrt{\frac{n+6}{6(n+2)}}$	$\sqrt{\frac{2(n+1)(n+6)}{3(n+2)(n+4)}}$
[21]	$\sqrt{\frac{(n+6)(n-2)}{3(n+4)(n-1)}}$	$\sqrt{\frac{n}{3(n-1)}}$	$-\sqrt{\frac{n(n+1)}{3(n+4)(n-1)}}$
[1]	$\sqrt{\frac{n(n+1)(n+6)}{2(n+4)(n-1)(n+2)}}$	$-\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}$	$\sqrt{\frac{2(n-2)}{(n-1)(n+4)(n+2)}}$

Table 13. $U_{O(n)}(1^2 1 1^2 1; v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	$[1^3]$
[0]	$\sqrt{\frac{2}{n(n-1)}}$	$\sqrt{\frac{2(n^2-4)}{3n(n-1)}}$	$\sqrt{\frac{n-2}{3n}}$
[2]	$\sqrt{\frac{n^2-4}{2n(n-1)}}$	$\sqrt{\frac{(n-4)^2}{6n(n-1)}}$	$-\sqrt{\frac{n+2}{3n}}$
$[1^2]$	$-\sqrt{\frac{n-2}{2(n-1)}}$	$\sqrt{\frac{n+2}{6(n-1)}}$	$-\sqrt{\frac{1}{3}}$

Table 14. $U_{O(n)}(1 1^2 1^2 1; v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]	$[1^3]$
[1]	$\frac{1}{n-1}$	$-\sqrt{\frac{n^2-4}{3(n-1)^2}}$	$\sqrt{\frac{2(n-2)}{3(n-1)}}$
[21]	$-\sqrt{\frac{n^2-4}{3(n-1)^2}}$	$\frac{2n-5}{3(n-1)}$	$\sqrt{\frac{2(n+2)}{9(n-1)}}$
$[1^3]$	$\sqrt{\frac{2(n-2)}{3(n-1)}}$	$\sqrt{\frac{2(n+2)}{9(n-1)}}$	$\frac{1}{3}$

coefficients of $O(n)$ in pairs so that we never require to introduce imaginary phase factors for Racah coefficients of $Sp(2m)$ after the replacement $n \to -2m$, and can always choose the phase factor η to be $+1$. For example, one can obtain Racah coefficients of $Sp(2m)$ given in table 5 from those of $O(n)$ listed in table 8.

Table 15. $U_{O(n)}(211^21;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]
[2]	$-\sqrt{\frac{n}{2(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$
$[1^2]$	$\sqrt{\frac{n-2}{2(n-1)}}$	$\sqrt{\frac{n}{2(n-1)}}$

Table 16. $U_{O(n)}(121^21;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]
[1]	$\frac{1}{n-1}$	$\sqrt{\frac{n(n-2)}{(n-1)^2}}$
[21]	$-\sqrt{\frac{n(n-2)}{(n-1)^2}}$	$\frac{1}{n-1}$

Table 17. $U_{O(n)}(2131^2;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[21]	[3]
[1]	$-\sqrt{\frac{2(n-2)}{3(n-1)}}$	$\sqrt{\frac{n+1}{3(n-1)}}$
[21]	$\sqrt{\frac{n+1}{3(n-1)}}$	$\sqrt{\frac{2(n-2)}{3(n-1)}}$

Table 18. $U_{O(n)}(211^31^2;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]
$[1^3]$	$\sqrt{\frac{n-3}{3(n-1)}}$	$\sqrt{\frac{2n}{3(n-1)}}$
[21]	$-\sqrt{\frac{2n}{3(n-1)}}$	$\sqrt{\frac{n-3}{3(n-1)}}$

Table 19. $U_{O(n)}(21^231;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[31]	[2]
[21]	$\sqrt{\frac{2}{n(n-1)}}$	$\sqrt{\frac{(n-2)(n+1)}{n(n-1)}}$
[1]	$-\sqrt{\frac{(n-2)(n+1)}{n(n-1)}}$	$\sqrt{\frac{2}{n(n-1)}}$

Table 20. $U_{O(n)}(21^21^31;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[211]	$[1^2]$
[21]	$-\sqrt{\frac{2(n-3)}{3(n-2)}}$	$\sqrt{\frac{n}{3(n-2)}}$
$[1^3]$	$\sqrt{\frac{n}{3(n-2)}}$	$\sqrt{\frac{2(n-3)}{3(n-2}}$

Table 21. $U_{\mathrm{O}(n)}(1^2121;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]
[2]	$-\sqrt{\frac{n}{2(n-1)}}$	$\sqrt{\frac{n-2}{2(n-1)}}$
$[1^2]$	$\sqrt{\frac{n-2}{2(n-1)}}$	$\sqrt{\frac{n}{2(n-1)}}$

Table 22. $U_{\mathrm{O}(n)}(11^221;\, v_{12}v_{23})$.

$v_{23}\backslash v_{12}$	[1]	[21]
[1]	$\frac{1}{n-1}$	$-\sqrt{\frac{n(n-2)}{(n-1)^2}}$
[21]	$\sqrt{\frac{n(n-2)}{(n-1)^2}}$	$\frac{1}{n-1}$

5. Conclusion

In this paper, we have given a formula for evaluating Racah coefficients of $\mathrm{O}(n)$ and $Sp(2m)$ from SDCs of Brauer algebras $D_f(n)$ by using the Schur–Weyl duality relation between $D_f(n)$ and $\mathrm{O}(n)$ or $Sp(2m)$. We found that there are two types of $\mathrm{O}(n)$ and $Sp(2m)$ Racah coefficients. The type-one Racah coefficients are n-independent, which are the same as those of $U(n)$; and $\mathrm{O}(n)$, and $Sp(2m)$ Racah coefficients are equal to each other with the same irreps in the coupling. The type-two Racah coefficients of $\mathrm{O}(n)$ and $Sp(2m)$ are usually n-dependent. Using the isomorphism between $B_f(O(n))$ and $B_f(Sp(2m))$, we can obtain $Sp(2m)$ Racah coefficients from those of $\mathrm{O}(n)$ by the replacement $n \to -2m$ and the conjugation of irreps. It should be noted that the results tabulated in this paper are all multiplicity-free in the coupling. The non-multiplicity-free type-one Racah coefficients of $U(n)$ given in [12] are also those of $\mathrm{O}(n)$ and $Sp(2m)$ of the same irreps, while those of type two can also be derived by using our method. The procedure to be taken for the multiplicity case is the same as that discussed in [19]. Finally, it should be pointed out that the Racah coefficients of quantum groups of B, C, and D types can also be derived from SDCs of Birman–Wenzl algebras $C_f(r, q)$. The SDCs of $C_f(r, q)$ can be evaluated by using the same method outlined in I, which will be discussed in the near future.

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